

The ideal relativistic spinning gas: polarization and spectra

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We study the physics of the ideal relativistic rotating gas at thermodynamical equilibrium and provide analytical expressions of the momentum spectra and polarization vector for the case of massive particles with spin 1/2 and 1. We show that the finite angular momentum \mathbf{J} entails an anisotropy in momentum spectra, with particles emitted orthogonally to \mathbf{J} having, on average, a larger momentum than along its direction. Unlike in the non-relativistic case, the proper polarization vector turns out not to be aligned with the total angular momentum with a non-trivial momentum dependence.

I. INTRODUCTION

The thermodynamics of rotating systems is a subject of interest mainly in astrophysics, where spinning objects are compact stars. Also in nuclear physics, there have been noteworthy applications to the problem of multifragmentation [1, 2]. In most those applications, either the non-relativistic approximation is used or the spin of particles is neglected, so that a full description of the relativistic case including spin degrees of freedom is still missing. However, recently, the microcanonical and grand-canonical partition function of an ideal relativistic quantum gas of particles with spin have been calculated [3] enforcing angular momentum conservation. Taking advantage of these results, in this paper we work out analytical expressions of the spectra and polarization of particles in a relativistic rotating gas with large (in \hbar units) angular momentum. From a phenomenological point of view, these calculations might be of interest for the physics of relativistic heavy ion collisions, where the formation of a system with a large intrinsic angular momentum has been envisaged [4]. We will confine ourselves to the case of Boltzmann statistics, leaving the quantum statistics case to future work.

The paper is organized as follows: in Sect. II we discuss the statistical mechanics of an ideal relativistic gas with fixed, large angular momentum and show the equivalence with a rigidly rotating system; in Sect. III we analyze more in detail the relation between angular momentum and angular velocity in the limit of low rotational speed; in Sect. IV we calculate the inclusive momentum spectra of particles; finally, in Sect. V we obtain the expressions of the polarization vectors of massive particles with spin 1/2 and 1.

Notation

In this paper we adopt the natural units, with $\hbar = c = K = 1$. Space-time linear transformations (translations, rotations, boosts) and $SL(2, \mathbb{C})$ transformations are written in serif font, e.g. R, L . Operators in Hilbert space will be denoted by an upper hat, e.g. \hat{R} . Unit vectors will be denoted with a smaller hat e.g. $\hat{\mathbf{p}}$. Even though the notation is unambiguous, we will make explicit mention of either possibility whenever confusion may arise.

II. ROTATING RELATIVISTIC GAS AT EQUILIBRIUM

In general, by rotating thermodynamical system we mean a system with fixed angular momentum in its rest frame. This is a good definition also in the relativistic case. It is well known that a classical system with non-vanishing intrinsic angular momentum can be at thermodynamical equilibrium only if the rotation is rigid [5], that is if a constant vector $\boldsymbol{\omega}$ exists such that the local collective velocity \mathbf{v} is:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} \quad (1)$$

We will show that this conclusion holds for the relativistic gas, with the obvious inequality:

$$\|\boldsymbol{\omega} \times \mathbf{x}\| < 1 \quad (2)$$

In this section, we will develop a proof based on an *ab initio* statistical mechanics calculation while a proof based on an extension of Landau's argument for relativistic systems is given in Appendix B.

In ref. [3], one of us derived the full expression of the microcanonical partition function, as well as its grand-canonical limit for large volumes, of a multi-species ideal relativistic gas with fixed angular momentum. In relativistic quantum mechanics, fixing the angular momentum means projecting onto irreducible states of the orthochronous Poincaré group $\text{IO}(1,3)^\dagger$ which, in fact, is a projection onto angular momentum states in the rest frame, where the total linear momentum vanishes. If, moreover, the intrinsic angular momentum is large (in \hbar units), it can be treated as a classical vector \mathbf{J} and the grand-canonical partition function reads [3]:

$$Z_J = \text{tr}\{\exp[(-\hat{H} + \mu\hat{Q})/T]P_{\mathbf{J}}P_V\} \\ = \frac{2J+1}{8\pi^2} \int_{|\phi|<\pi} d^3\phi e^{i\phi\cdot\mathbf{J}} \exp\left[\sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p e^{-\varepsilon_j/T} e^{-i\phi\cdot(\mathbf{x}\times\mathbf{p})} \text{tr} D^{S_j}(\mathbf{R}_{\hat{\phi}}(\phi))\right] \quad (3)$$

where T is the temperature and V the volume; ε_j is the energy, λ_j the fugacity and S_j the spin of the j th particle species; $D^{S_j}(\mathbf{R}_{\hat{\phi}}(\phi))$ is the matrix of the irreducible representation S_j of $\text{SU}(2)$ transformation \mathbf{R} with axis $\hat{\phi}$ and angle $\phi = \|\phi\|$. The operator \hat{H} is the hamiltonian, \hat{Q} is a generic conserved charge operator and μ the relevant chemical potential; the operator $P_{\mathbf{J}}$ is the projector onto a fixed angular momentum while:

$$P_V = \sum_{h_V} |h_V\rangle\langle h_V|$$

is the projector onto localized states $|h_V\rangle$, which form a complete set of quantum states for the system in the finite region V . This projector is needed in eq. (3) for the trace operation to be a properly defined one, i.e. involving a basis of the *full* Hilbert space [3].

The formula (3) applies to a system with large V and J and the integrand function turns out to be significant only in a region where $\phi = \|\phi\| \ll 1$. Since both J and V are large, it is then possible to make a saddle-point expansion of the integral (3). To do this, we first have to continue the integration variables to the complex plane and, in view of the spherical symmetry of the domain, the obvious choice is to take the spherical coordinates of the vector ϕ , namely its magnitude ϕ along with its polar and azimuthal angles. Then, we have to solve the complex vector equation:

$$\nabla_{\phi} \left[i\phi \cdot \mathbf{J} + \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \text{tr} D^{S_j}(\mathbf{R}_{\hat{\phi}}(\phi)) e^{-\varepsilon_j/T} e^{-i\phi\cdot(\mathbf{x}\times\mathbf{p})} \right] = 0 \quad (4)$$

leading to:

$$\mathbf{J} = \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \text{tr} D^{S_j}(\mathbf{R}_{\hat{\phi}}(\phi)) (\mathbf{x} \times \mathbf{p}) e^{-\varepsilon_j/T} e^{-i\phi\cdot(\mathbf{x}\times\mathbf{p})} \\ + \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \left[i\nabla_{\phi} \text{tr} D^{S_j}(\mathbf{R}_{\hat{\phi}}(\phi)) \right] e^{-\varepsilon_j/T} e^{-i\phi\cdot(\mathbf{x}\times\mathbf{p})} \\ \equiv \mathbf{L}(\phi) + \mathbf{S}(\phi) \quad (5)$$

the reason for naming the two integral terms as $\mathbf{L}(\phi)$ and $\mathbf{S}(\phi)$ will become clear later on. We first observe that, if the equation (4) is solved by $\phi, \hat{\phi}$, then also $-\phi^*, \hat{\phi}^*$ is a solution. This can be easily checked by taking into account that the trace depends only on ϕ and can be written as a sum of exponentials $\exp[in\phi]$, being n an integer. Therefore, we will look for one solution enforcing $\phi = -\phi^*$ and $\hat{\phi} = \hat{\phi}^*$, that is with a real unit vector $\hat{\phi}$ and an imaginary magnitude ϕ .

Assuming that one solution exists and defining the real vector $\boldsymbol{\omega} \equiv -iT\phi$, we can write the grand-canonical partition function at the lowest order of the saddle-point expansion as:

$$Z_J \propto \exp[-\boldsymbol{\omega} \cdot \mathbf{J}/T] \exp\left[\sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) e^{-\varepsilon_j/T} e^{\boldsymbol{\omega}\cdot(\mathbf{x}\times\mathbf{p})/T}\right] \quad (6)$$

The function:

$$Z_{\omega} = \exp\left[\sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) e^{-\varepsilon_j/T} e^{\boldsymbol{\omega}\cdot(\mathbf{x}\times\mathbf{p})/T}\right] \quad (7)$$

is the partition function of an ideal relativistic gas (in the Boltzmann limit) rotating with an angular velocity $\boldsymbol{\omega}$. This can be shown explicitly for the spinless case by dividing the gas into small cells with volume $\Delta^3 x$ and uniform velocity (1) and calculating the relevant grand-canonical partition function in the formalism of relativistic thermodynamics [6]:

$$Z_{\text{cell}} = \exp \left[\sum_j \frac{\lambda_j}{(2\pi)^3} \Delta^3 x \int d^3 p e^{-\beta \cdot p} \right] \quad (8)$$

where β is the temperature four-vector:

$$\beta = \frac{1}{T_0 \sqrt{1 - v^2}} (1, \mathbf{v}) = \frac{1}{T_0 \sqrt{1 - v^2}} (1, \boldsymbol{\omega} \times \mathbf{x}) \quad (9)$$

T_0 being the *local* temperature, i.e. the temperature measured by a thermometer moving along with the cell. Note that with this equation, we are tacitly assuming that the partition function, which is a Lorentz-invariant quantity, in the accelerated rotating cell is the same as that in an inertial frame with its instantaneous origin and velocity¹. This is consistent with the locality hypothesis [7] and the transformation law of energy in a rotating frame [7]:

$$E' = \gamma(E - \boldsymbol{\omega} \cdot \mathbf{L}) = \gamma(E - \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{P})) \quad (10)$$

being $\gamma = 1/\sqrt{1 - \|\boldsymbol{\omega} \times \mathbf{x}\|^2}$. Therefore, with the identification:

$$T = T_0 \sqrt{1 - v^2} = T_0 \sqrt{1 - \|\boldsymbol{\omega} \times \mathbf{x}\|^2} \quad (11)$$

the partition function of the cell becomes:

$$Z_{\text{cell}} = \exp \left[\sum_j \frac{\lambda_j}{(2\pi)^3} \Delta^3 x \int d^3 p e^{-\varepsilon_j/T} e^{\mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{x})/T} \right] \quad (12)$$

The total partition function of the supposedly independent cells can now be obtained by multiplying expressions (12) for all cells and going to the limit of infinitesimal cells, implying:

$$Z_{\omega} = \exp \left[\sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3 x \int d^3 p e^{-\varepsilon_j/T} e^{\boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T} \right] \quad (13)$$

which is the (7) for spinless particles.

Therefore, unlike in the non-relativistic case, the proper temperature T_0 in a spinning relativistic system at equilibrium depends on the distance from the rotation axis, an effect pointed out by Israel [8]. However, the thermal equilibrium is characterized by a uniform temperature T as measured by the observer; another proof of this statement based on simple arguments can be found in Appendix B.

The partition function Z_{ω} defines an ensemble, that we can define as *rotational grand-canonical* where the angular velocity is fixed and the intrinsic angular momentum \mathbf{J} can fluctuate. On the other hand, the grand-canonical partition function with fixed intrinsic angular momentum \mathbf{J} defines an ensemble that can be called *micro-rotational grand-canonical*. The above definitions are inspired of the more familiar canonical-microcanonical duality, according to whether T or E are fixed.

Pursuing the analogy, it is expected that these two ensembles become equivalent in the thermodynamic limit as far as the calculation of first-order moments of statistical distributions is concerned. In fact, neglecting the constant small factors multiplying Z_J , it can be seen from (6) that $\log Z_{\omega}$ is essentially the Legendre transform of $\log Z_J$ with respect to the total angular momentum \mathbf{J} :

$$\log Z_{\omega} = \log Z_J + \frac{\boldsymbol{\omega} \cdot \mathbf{J}}{T} = \log Z_J + \mathbf{J} \cdot \frac{\partial}{\partial \mathbf{J}} \log Z_J \quad (14)$$

¹ Note that thermodynamic equilibrium is possible in a rotating cell because its acceleration is stationary

The saddle-point equation (5) for $\phi = i\omega/T$ is simply the inverse Legendre transformation of (14):

$$\begin{aligned} \mathbf{J} &= \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3\mathbf{x} \int d^3\mathbf{p} \operatorname{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) (\mathbf{x} \times \mathbf{p}) e^{-\varepsilon_j/T} e^{\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \\ &+ \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3\mathbf{x} \int d^3\mathbf{p} \left[\frac{\partial}{\partial \omega/T} \operatorname{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \right] \hat{\omega} e^{-\varepsilon_j/T} e^{\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \\ &= \mathbf{L}(\omega/T) + \mathbf{S}(\omega/T) = \frac{\partial}{\partial \omega/T} \log Z_\omega \end{aligned} \quad (15)$$

Therefore, the saddle point equation (15) relates angular velocity ω and angular momentum \mathbf{J} and applies to relativistic as well as non-relativistic gases. It expresses the conservation of angular momentum, in the sense that the total angular momentum in the rotational grand-canonical ensemble on the right hand side, where the two contributions of orbital and spin can be clearly identified, equals the initial conserved total angular momentum \mathbf{J} on the left hand side. It is worth pointing out that for macroscopic systems, the contribution of the orbital angular momentum is predominant. This can be seen from eq. (15) noting that the integral defining the total orbital angular momentum is scaled by a factor $\|\mathbf{x} \times \mathbf{p}\|$ with respect to the spin angular momentum. Since this is generally much larger than 1, for macroscopic distances and not too low temperatures, we usually have $\|\mathbf{L}\| \gg \|\mathbf{S}\|$ and the spin angular momentum can be neglected. Yet, for not too large systems, the spin angular momentum contribution can be sizeable. There is also a remarkable difference between them concerning their dependence on the size of the system: the spin angular momentum is properly *extensive*, i.e. it increases linearly with the volume, while the orbital angular momentum is not because of the factor $\|\mathbf{x} \times \mathbf{p}\|$. The lack of extensivity of rotating systems makes the thermodynamic limit essentially irrelevant and dictates a finite-size treatment. From a completely equivalent point of view of the comoving frame, we can state that the rotating system is non-extensive because of the presence of a long-range centrifugal potential.

Finally, it can be shown that in the rotational grand-canonical ensemble each microstate with total energy E , total charge Q and total angular momentum \mathbf{J} has a probability (in the frame where $\langle \mathbf{P} \rangle = 0$):

$$p(T, \mu, \omega) \propto \exp[-E/T + \mu Q/T + \omega \cdot \mathbf{J}/T] \quad (16)$$

and, consequently, rotational grand-canonical partition function can be written as:

$$Z_\omega = \operatorname{tr}\{\exp[(-\hat{H} + \mu\hat{Q} + \omega \cdot \hat{\mathbf{J}})/T] P_V\} \quad (17)$$

where $\hat{\mathbf{J}}$ is the total angular momentum operator. This statement holds provided that $\omega/T \ll 1$ so that terms of the order $(\omega/T)^2$ and higher can be neglected; only in this case is the formula (17) equivalent to (7). Indeed, working out the trace (17) for an ideal gas involves the calculation of matrix elements such as:

$$\sum_\sigma \langle p, \sigma | \exp[\omega \cdot \hat{\mathbf{J}}/T] P_V | p, \sigma \rangle = \sum_\sigma \langle p, \sigma | \hat{\mathbf{R}}_{\hat{\omega}}(i\omega/T) P_V | p, \sigma \rangle \quad (18)$$

where $|p, \sigma\rangle$ are single particle states with four-momentum p and spin projection σ (see Sect. 5 for extended discussion) and the rotation of an imaginary angle $i\omega/T$ has been introduced. The above single-particle trace has been calculated in ref. [3]:

$$\sum_\sigma \langle p, \sigma | \hat{\mathbf{R}}_{\hat{\omega}}(i\omega/T) P_V | p, \sigma \rangle = \frac{1}{(2\pi)^3} \int_V d^3\mathbf{x} e^{i\mathbf{x} \cdot (\mathbf{p} - \mathbf{R}_{\hat{\omega}}(i\omega/T)^{-1} \mathbf{p})} \operatorname{tr} D^S(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \langle 0 | P_V | 0 \rangle \quad (19)$$

The vacuum expectation value of P_V is immaterial and also becomes 1 in the large volume limit. Now, since:

$$\mathbf{R}_{\hat{\omega}}(\psi) \mathbf{v} = \mathbf{v} \cos \psi + (\hat{\mathbf{n}} \times \mathbf{v}) \sin \psi + (1 - \cos \psi) \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \quad (20)$$

we have:

$$\mathbf{x} \cdot (\mathbf{p} - \mathbf{R}_{\hat{\omega}}(i\omega/T)^{-1} \mathbf{p}) \simeq i \frac{\omega}{T} \mathbf{x} \cdot (\omega \times \mathbf{p})$$

only if ω/T is sufficiently smaller than 1, so that cosine terms in (20) can be approximated with 1 and sine terms with their argument. In this case, eq. (19) turns into, for large V :

$$\sum_\sigma \langle p, \sigma | \hat{\mathbf{R}}_{\hat{\omega}}(i\omega/T) P_V | p, \sigma \rangle \simeq \frac{1}{(2\pi)^3} \int_V d^3\mathbf{x} e^{\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \operatorname{tr} D^S(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \quad (21)$$

It is then straightforward to show that the final expression of (17) is just (7).

III. MOMENTS OF INERTIA

In this section, we will study more in detail the saddle-point equation (5) or its equivalent form (15). This is a vector equation in $\boldsymbol{\omega}$, whose solution depends, besides T and \mathbf{J} , on the shape and size of the system.

If the total angular momentum \mathbf{J} is a symmetry axis of V , then \mathbf{J} and $\boldsymbol{\omega}$ must be parallel (see Fig. 1). This can be proved by applying an arbitrary rotation of an angle ψ around \mathbf{J} , let $\mathbf{R}_{\mathbf{J}}(\psi)$, to both sides of eq. (15). Since $\mathbf{R}(\mathbf{x} \times \mathbf{p}) = \mathbf{R}\mathbf{x} \times \mathbf{R}\mathbf{p}$, defining new integration variables $\mathbf{x}' = \mathbf{R}\mathbf{x}$ and $\mathbf{p}' = \mathbf{R}\mathbf{p}$, taking into account that $\boldsymbol{\omega} \cdot \mathbf{R}^{-1}(\mathbf{x}' \times \mathbf{p}') = \mathbf{R}\boldsymbol{\omega} \cdot (\mathbf{x}' \times \mathbf{p}')$ and being both momentum and spacial domains invariant by rotation around \mathbf{J} , we can conclude that, if $\boldsymbol{\omega}$ is a solution of the eq. (4), $\mathbf{R}_{\mathbf{J}}(\psi)\boldsymbol{\omega}$ is also a solution *for any* ψ . Therefore, $\mathbf{R}_{\mathbf{J}}(\psi)\boldsymbol{\omega}$ should coincide with $\boldsymbol{\omega}$ for the equation to be well behaved. If this is the case, $\boldsymbol{\omega}$ is parallel to \mathbf{J} and the orbital term $\mathbf{L}(\boldsymbol{\omega}/T)$ should be also parallel to $\boldsymbol{\omega}$. In this case, the eq. (15) essentially reduces to a scalar one:

$$J = \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \operatorname{tr} D^{S_j}(\mathbf{R}\boldsymbol{\omega}(i\omega/T)) \hat{\mathbf{J}} \cdot (\mathbf{x} \times \mathbf{p}) e^{-\varepsilon_j/T} e^{\omega \hat{\mathbf{J}} \cdot (\mathbf{x} \times \mathbf{p})/T} \\ + \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \left[\frac{\partial}{\partial \omega/T} \operatorname{tr} D^{S_j}(\mathbf{R}\boldsymbol{\omega}(i\omega/T)) \right] e^{-\varepsilon_j/T} e^{\omega \hat{\mathbf{J}} \cdot (\mathbf{x} \times \mathbf{p})/T} \quad (22)$$

which can be solved numerically.

Let us now introduce a length R related to the size of the system; it can be, e.g. the maximal distance of a point of the set V from the rotation axis. An analytic solution of the general equation (15) can be obtained for small values of $\omega R p/T$. In fact, under this circumstance, the exponential can be approximated as:

$$\exp[\boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T] \simeq 1 + \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T \quad (23)$$

This condition leads to different requirements for a non-relativistic and an ultrarelativistic gas (being $p = \mathcal{O}(\sqrt{mT})$ with $m \gg T$ and $p = \mathcal{O}(T)$ respectively), yet both imply that $\omega R \ll 1$. If approximation (23) applies, the two terms \mathbf{L} and \mathbf{S} reduce to, at the first order in ω :

$$\mathbf{L} \simeq \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p (2S_j + 1) (\mathbf{x} \times \mathbf{p}) \frac{\boldsymbol{\omega}}{T} \cdot (\mathbf{x} \times \mathbf{p}) e^{-\varepsilon_j/T} \\ \mathbf{S} \simeq \sum_j \frac{\lambda_j}{(2\pi)^3} \int_V d^3x \int d^3p \frac{S_j(S_j + 1)(2S_j + 1)}{3} e^{-\varepsilon_j/T} \frac{\boldsymbol{\omega}}{T} \quad (24)$$

By expanding the scalar and vector products in the integral of the orbital angular momentum and carrying out simple algebraic calculations, one obtains:

$$L_i = \sum_{k=1}^3 \sum_j \langle n_j \rangle \frac{\langle p_j^2 \rangle}{3T} \int_V d^3x (r^2 \delta_{ik} - x_i x_k) \omega_k \quad (25)$$

where $r^2 = \sum_{i=1}^3 x_i^2$ and:

$$\langle n_j \rangle = \frac{\lambda_j (2S_j + 1)}{(2\pi)^3} \int d^3p e^{-\varepsilon_j/T} \quad (26)$$

is the mean density of the species j in the Boltzmann approximation and without angular momentum constraint, while:

$$\langle p_j^2 \rangle = \frac{1}{\langle n_j \rangle} \frac{\lambda_j (2S_j + 1)}{(2\pi)^3} \int d^3p p^2 e^{-\varepsilon_j/T} \quad (27)$$

is the mean squared momentum of the particle j under the same approximations.

The relation between \mathbf{L} and $\boldsymbol{\omega}$ (25) resembles the classical linear relation between angular velocity and angular momentum defining the inertia tensor \mathbf{I} . Indeed, in the non-relativistic limit, this is precisely what one gets from (25) because $\langle p_j^2 \rangle = 3m_j T$ and, consequently:

$$L_i = \sum_{k=1}^3 \sum_j m_j \langle n_j \rangle \int_V d^3x (r^2 \delta_{ik} - x_i x_k) \omega_k = \sum_{k=1}^3 I_{ik} \omega_k \quad (28)$$

Hence the expression in (25):

$$I_{ik} = \sum_j \langle n_j \rangle \frac{\langle p_j^2 \rangle}{3T} \int_V d^3x (r^2 \delta_{ik} - x_i x_k) \quad (29)$$

turns out to be the right generalization of the inertia tensor for a relativistic gas. Like in the classical case, if the system (which is homogeneous for small ω) is symmetric around its rotation axis, then \mathbf{L} is parallel to $\boldsymbol{\omega}$ and the proportionality constant is just the moment of inertia with respect to that axis.

Unlike in the classical case, for non-macroscopic systems there is also a potentially non-negligible spin contribution. In this case, the full relation between \mathbf{J} and $\boldsymbol{\omega}$ reads:

$$J_i = \sum_{k=1}^3 \sum_j \langle n_j \rangle \left[\frac{\langle p_j^2 \rangle}{3T} \int_V d^3x (r^2 \delta_{ik} - x_i x_k) + V \frac{S_j(S_j + 1)}{3T} \delta_{ik} \right] \omega_k \quad (30)$$

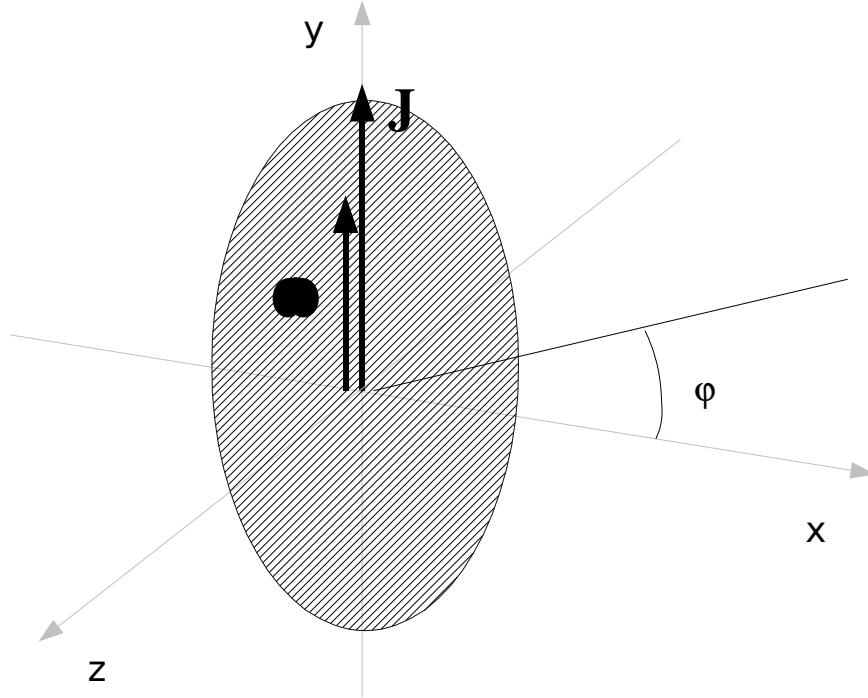


FIG. 1: For an axially symmetric system around the angular momentum direction, $\boldsymbol{\omega}$ is parallel to \mathbf{J} . Also shown the coordinates chosen to describe spectra in Sect. IV.

IV. SPECTRA AND ANISOTROPIES

The impact of a finite angular momentum on particle spectra in a relativistic gas was considered by Hagedorn [9] who pointed out that a large value of the intrinsic angular momentum involves a peculiar anisotropy. Here, we will provide the most general expressions of the momentum spectrum of particles in an equilibrated rotating system with fixed (classical) angular momentum \mathbf{J} . This can be obtained directly, for the j th particle species, from the partition function (3):

$$\left\langle \frac{dn_j}{d^3p} \right\rangle = \frac{1}{Z_J} \text{tr} \left\{ \frac{d\hat{n}_j}{d^3p} \exp[(-\hat{H} + \mu\hat{Q})/T] P_J P_V \right\}$$

where $d\hat{n}_j/d^3p$ is the momentum spectrum *operator*. The above expression can be rewritten as the functional derivative with respect to $\alpha(\mathbf{p})$ of:

$$Z_J[\alpha(\mathbf{p})] = \text{tr} \left\{ \exp \left[(-\hat{H} + \mu\hat{Q})/T + \int d^3p \alpha(\mathbf{p}) \frac{d\hat{n}_j}{d^3p} \right] P_J P_V \right\} \quad (31)$$

in $\alpha(\mathbf{p}) = 0$. Since, for an ideal gas:

$$\hat{H} = \sum_j \int d^3\mathbf{p} \varepsilon_j(\mathbf{p}) \frac{d\hat{n}_j}{d^3\mathbf{p}}$$

the functional $Z_J[\alpha(\mathbf{p})]$ can be simply obtained from eq. (3) replacing ε_j/T with $\varepsilon_j/T + \alpha(\mathbf{p})$. The functional derivative then turns out to be:

$$\begin{aligned} \left\langle \frac{dn_j}{d^3\mathbf{p}} \right\rangle &= \frac{\delta}{\delta\alpha(\mathbf{p})} \log Z_J[\alpha(\mathbf{p})] \Big|_{\alpha(\mathbf{p})=0} = \frac{1}{Z_J} \frac{2J+1}{8\pi^2} \frac{\lambda_j}{(2\pi)^3} \int_V d^3\mathbf{x} e^{-\varepsilon_j/T} \int_{|\phi|<\pi} d^3\phi e^{i\phi \cdot (\mathbf{J} - \mathbf{x} \times \mathbf{p})} \text{tr} D^{S_j}(\mathbf{R}_\phi(\phi)) \\ &\times \exp \left[\sum_k \frac{\lambda_k}{(2\pi)^3} \int_V d^3\mathbf{x} \int d^3\mathbf{p} e^{-\varepsilon_k/T} e^{-i\phi \cdot (\mathbf{x} \times \mathbf{p})} \text{tr} D^{S_k}(\mathbf{R}_\phi(\phi)) \right] \end{aligned} \quad (32)$$

Similarly to eq. (3), it is possible to make a saddle-point expansion of the above integral. If the orbital angular momentum of the particle is much lower than the total angular momentum, i.e. $\|\mathbf{x} \times \mathbf{p}\| \ll J$, the former can be neglected in the saddle-point equation which is then essentially the same as that for the partition function, eq. (4). This condition amounts to take $\exp[-i\phi \cdot (\mathbf{x} \times \mathbf{p})]$ as a constant factor and it is usually met except for momenta larger than J/R , R being the size of the system. In this case, the formula (32) becomes, at the leading order:

$$\left\langle \frac{dn_j}{d^3\mathbf{p}} \right\rangle = \frac{\lambda_j}{(2\pi)^3} \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \int_V d^3\mathbf{x} e^{-\varepsilon_j/T} e^{\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \quad (33)$$

which is, of course, the spectrum in the rotational grand-canonical ensemble in the Boltzmann limit.

It is of some interest to derive the shape of the spectra in cylindrical momentum coordinates $p_T = \sqrt{p_x^2 + p_y^2}$, azimuthal angle φ and rapidity $y = 1/2 \log[(p_z + \varepsilon)/(\varepsilon - p_z)]$ when the angular momentum of the system is directed along the y axis, i.e. $\mathbf{J} = J\hat{\mathbf{y}}$ and the system is rotationally symmetric around \mathbf{J} . Because of this symmetry one has $\omega = \omega\hat{\mathbf{y}}$ and the spectrum (33) reads:

$$\left\langle \frac{dn_j}{d^3\mathbf{p}} \right\rangle = \frac{\lambda_j}{(2\pi)^3} \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \int_V d^3\mathbf{x} e^{-\varepsilon_j/T} e^{\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \quad (34)$$

In the new coordinates p_T, φ, y the spectrum reads:

$$\left\langle \frac{dn_j}{p_T dp_T d\varphi dy} \right\rangle = \frac{\lambda_j \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T))}{(2\pi)^3} \int_V d^3\mathbf{x} m_T \cosh y \exp[-m_T \cosh y/T + \mathbf{p}_T \cdot (\omega \times \mathbf{x})_\perp/T + m_T (\omega \times \mathbf{x})_\parallel \sinh y/T] \quad (35)$$

where $m_T = \sqrt{p_T^2 + m_j^2}$, \parallel labels the longitudinal projection along the z axis and \perp the transverse projection onto the xy plane.

With successive integrations of eq. (35) one obtains:

$$\left\langle \frac{dn_j}{p_T dp_T d\varphi} \right\rangle = \frac{\lambda_j \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T))}{4\pi^3} \int_V d^3\mathbf{x} \frac{1}{\sqrt{1 - (\omega \times \mathbf{x})_\parallel^2}} m_T K_1 \left(m_T \sqrt{1 - (\omega \times \mathbf{x})_\parallel^2}/T \right) \exp[\mathbf{p}_T \cdot (\omega \times \mathbf{x})_\perp/T] \quad (36)$$

and:

$$\left\langle \frac{dn_j}{p_T dp_T} \right\rangle = \frac{\lambda_j \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T))}{2\pi^2} \int_V d^3\mathbf{x} \frac{1}{\sqrt{1 - (\omega \times \mathbf{x})_\parallel^2}} m_T K_1 \left(m_T \sqrt{1 - (\omega \times \mathbf{x})_\parallel^2}/T \right) I_0(p_T \|(\omega \times \mathbf{x})_\perp\|/T) \quad (37)$$

where K_1 and I_0 are McDonald and modified Bessel functions respectively. It is also possible to calculate the mean multiplicity by integrating (37), which yields:

$$\langle n_j \rangle = \frac{\lambda_j}{2\pi^2} \text{tr} D^{S_j}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \int_V d^3\mathbf{x} \frac{m^2 T}{1 - \|\omega \times \mathbf{x}\|^2} K_2 \left(m \sqrt{1 - \|\omega \times \mathbf{x}\|^2}/T \right) \quad (38)$$

The formulae (35)-(38) yield as limiting cases for $\omega = 0$ the well known ones for the relativistic ideal Boltzmann gas.

It is worth pointing out that the spectrum (35) features a non-trivial dependence on the momentum azimuthal angle φ strongly sensitive to the ratio ω/T . Introducing cylindrical coordinates r, Φ, z for the vector \mathbf{x} and taking into account that $\boldsymbol{\omega} = \omega \hat{\mathbf{j}}$, eq. (36) can be written as:

$$\left\langle \frac{dn_j}{p_T dp_T d\varphi} \right\rangle = \frac{\lambda_j \text{tr} D^{S_j}(\mathbf{R}_{\hat{\boldsymbol{\omega}}}(i\omega/T))}{4\pi^3} \int_V d^3x \frac{1}{\sqrt{1 - (\omega r \cos \Phi)^2}} m_T K_1 \left(m_T \sqrt{1 - (\omega r \cos \Phi)^2} / T \right) \exp[p_T \omega z \cos \varphi / T] \quad (39)$$

which shows that particles emitted along the momentum axis (with $\varphi = \pi/2$) have, on average, a lower momentum than those emitted orthogonally to it. Classically, this can be understood as an effect of centrifugal force.

The (39) is a periodic function in the momentum azimuthal angle φ , thus it can be Fourier expanded. Borrowing a notation commonly used in relativistic heavy ion physics, we will denote by $2v_k$ the coefficients of this expansion, i.e. setting:

$$\left\langle \frac{dn_j}{p_T dp_T d\varphi} \right\rangle = \frac{1}{2\pi} \left\langle \frac{dn_j}{p_T dp_T} \right\rangle \sum_{k=0}^{\infty} 2v_{2k} \cos 2k\varphi \quad (40)$$

with $v_0 \equiv 1$ we obtain:

$$v_{2k} = \frac{\int_0^{2\pi} \left\langle \frac{dn_j}{p_T dp_T d\varphi} \right\rangle \cos 2k\varphi}{\int_0^{2\pi} \left\langle \frac{dn_j}{p_T dp_T d\varphi} \right\rangle} \quad (41)$$

and, by using (39)

$$v_{2k} = \frac{\int_V d^3x \frac{1}{\sqrt{1 - (\omega r \cos \Phi)^2}} m_T K_1 \left(m_T \sqrt{1 - (\omega r \cos \Phi)^2} / T \right) I_{2k}(p_T z \omega / T)}{\int_V d^3x \frac{1}{\sqrt{1 - (\omega r \cos \Phi)^2}} m_T K_1 \left(m_T \sqrt{1 - (\omega r \cos \Phi)^2} / T \right) I_0(p_T z \omega / T)} \quad (42)$$

which shows that the Fourier coefficients are bound between 0 and 1. For low p_T , the ratio between modified Bessel functions is such that $I_{2k}/I_0 \propto p_T^{2k}$, while at large p_T values, the ratio of modified Bessel functions tends to 1 and, as a consequence, all $v_{2k} \rightarrow 1$.

V. POLARIZATION

If a gas has a net electric charge, an imbalance in the mean multiplicities of positive and negative particles is implied. Similarly, if a gas at thermodynamical equilibrium has a non-vanishing angular momentum (hence it is rigidly rotating, as we have seen in Sect. II) particles should have a net polarization along the direction of angular momentum. From the viewpoint of the comoving observer in the rotating frame, the hamiltonian in the rotating frame has a spin-rotation coupling term [7, 10]². For the non-relativistic case, the calculation is straightforward as the polarization vector is the same in the observer and in the comoving frame, hence it can be carried out in the latter where the single-particle hamiltonian reads [10]:

$$\hat{h}_{\text{rot}} = \hat{h}_{\text{obs}} - \boldsymbol{\omega} \cdot \hat{\mathbf{j}} \quad (43)$$

being $\hat{\mathbf{j}}$ the *total* (orbital + spin) angular momentum operator of the particle. In fact, the problem we want to solve here is to calculate the polarization vector for a *relativistic* gas.

This problem is more difficult than for charge, energy or momentum because angular momentum is not a generator of an abelian group and this adds some complication. Particularly, from the point of view of statistical mechanics, one has to deal, as we will shortly see, with a spin density matrix with non-vanishing off-diagonal elements, unlike for momenta. From eq. (17), the density operator in the rotational grand-canonical ensemble reads:

$$\hat{\rho}_{\omega} = \frac{1}{Z_{\omega}} \exp[(-\hat{H} + \mu \hat{Q} + \boldsymbol{\omega} \cdot \hat{\mathbf{j}})/T] \mathcal{P}_V \quad (44)$$

² This phenomenon is well known for radio waves [7]

while in the micro-rotational grand-canonical ensemble, from eq. (3):

$$\hat{\rho}_J = \frac{1}{Z_J} \exp[(-\hat{H} + \mu\hat{Q})/T] \mathbf{P}_J \mathbf{P}_V \quad (45)$$

\mathbf{P}_J being the quantum projector onto states with definite angular momentum. As argued in Sect. II, the two ensembles are equivalent for large J and V provided that ω/T is sufficiently smaller than 1. Henceforth, we will confine our attention to this case and we will work in the rotational grand-canonical ensemble described by (44).

In the Boltzmann limit of the ideal relativistic quantum gas, all particles can be handled as independent distinguishable objects. Thus, the density operator (44) factorize and we can calculate the polarization by considering the single-particle density operator:

$$\hat{\rho}_\omega = \frac{1}{z_\omega} \exp[(-\hat{h} + \mu\hat{q} + \boldsymbol{\omega} \cdot \hat{\mathbf{j}})/T] \mathbf{P}_V \quad (46)$$

where z_ω is the single particle partition function:

$$z_\omega = \frac{\lambda}{(2\pi)^3} \int_V d^3\mathbf{x} \int d^3\mathbf{p} e^{-\varepsilon/T} e^{\boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T} \text{tr} D^S(\mathbf{R}_{\hat{\omega}}(i\omega/T)) \quad (47)$$

ensuring the correct normalization $\text{tr} \rho_\omega = 1$ (this in fact requires $\omega/T \ll 1$, see discussion at the end of Sect. II).

The density operator restricted to the spin degrees of freedom is a function of momentum and reads:

$$\hat{\rho}_\omega(p) = \frac{1}{\frac{\lambda}{(2\pi)^3} \int_V d^3\mathbf{x} e^{-\varepsilon/T} e^{\boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T} \text{tr} D^S(\mathbf{R}_{\hat{\omega}}(i\omega/T))} \exp[(-\hat{h} + \mu\hat{q} + \boldsymbol{\omega} \cdot \hat{\mathbf{j}})/T] \mathbf{P}_V \quad (48)$$

The polarization is, by definition, the trace of the density matrix $\rho(p)$ multiplied by a suitable spin operator. In non-relativistic quantum mechanics, this is obviously the spin vector operator $\hat{\mathbf{S}}$ and the polarization can be obtained straightforwardly:

$$\mathbf{\Pi} = \text{tr}[\hat{\mathbf{S}} \hat{\rho}_\omega(p)] = \frac{\sum_{n=-S}^S n e^{n\omega/T}}{\sum_{n=-S}^S e^{n\omega/T}} \hat{\omega} \quad (49)$$

which turns out to be momentum-independent. In relativistic quantum mechanics the proper generalization of the spin angular momentum is the Pauli-Lubanski vector multiplied by $1/m$:

$$\widehat{W}_\mu = -(1/2) \sum_{\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \hat{J}^{\nu\rho} \hat{P}^\sigma \quad (50)$$

which in fact reduces to the spin operator in the particle rest frame; $\hat{J}^{\nu\rho}$ are the generators of the Lorentz group. The polarization is thus a four-vector $\mathbf{\Pi}$:

$$\mathbf{\Pi}(p) = \frac{1}{m} \text{tr}_p(\widehat{W} \hat{\rho}(p)) \quad (51)$$

where the trace is to be calculated by summing only over spin degrees of freedom keeping the four-momentum p fixed. This four-vector $\mathbf{\Pi}(p)$ has vanishing time component in the particle rest frame, as it is apparent from (50). Before working out eq. (51), we should introduce some important notions about the construction of physical states. We will stick to the notation of ref. [11].

As it is well known, the Pauli-Lubanski vector fulfills the commutation relations:

$$\begin{aligned} [\widehat{W}_\mu, \widehat{P}_\nu] &= 0 \\ [\widehat{W}_\mu, \widehat{W}_\nu] &= -i \sum_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \widehat{W}^\rho \widehat{P}^\sigma \\ \widehat{W} \cdot \widehat{P} &= 0 \end{aligned} \quad (52)$$

Hence, if the ket $|p\rangle$ is an eigenvector of \widehat{P} , so is $\widehat{W}|p\rangle$. The restriction of \widehat{W} to the eigenspace labelled by four-momentum P is defined as $\widehat{W}(p)$. Since $\widehat{W}(p) \cdot p = 0$, this four-vector operator can be decomposed onto three

orthonormal spacelike four-vectors $n_1(p), n_2(p), n_3(p)$ which form a basis of the Minkowski space with the unit vector $\hat{p} = p/\sqrt{p^2}$:

$$\widehat{W}(p) = \sum_{i=1}^3 \widehat{W}_i(p) n_i(p) \quad (53)$$

It can be shown that the operators:

$$\widehat{S}_i(p) = \widehat{W}_i(p)/m \quad (54)$$

form an $SU(2)$ algebra and are the actual relativistic generalization of the spin angular momentum. The third component $\widehat{S}_3(p)$ can be diagonalized along with $\widehat{S}^2 = -\widehat{W}^2/m^2$ which is a Casimir of the full group $IO(1,3)^\dagger$, with corresponding eigenvalues σ and $S(S+1)$. With a suitable choice of $n_i(p)$, i.e.:

$$n_i(p) = [p] e_i \quad [p] \equiv R_3(\varphi) R_2(\theta) L_3(\xi) \quad (55)$$

e_i being the unit vectors of spacial axes and $[p]$ being a Lorentz transformation bringing the timelike vector $p_0 = (m, 0, 0, 0)$ into the four-momentum p with polar coordinates ξ, θ, φ ; the eigenvalue λ has the physical meaning of the component of intrinsic angular momentum in the rest frame along the direction of particle momentum \mathbf{p} . Thus, with the choice (55), λ is the helicity in the rest frame and J is, by definition, the spin of the particle. Since, from eqs. (54), (53) and (50) $\widehat{S}_i(p_0) = \widehat{J}_i$, the spin operators in the rest frame coincide with the generators of the rotation groups. Finally, single particle states will be written as $|p, \sigma\rangle$ with:

$$\widehat{P}|p, \sigma\rangle = p|p, \sigma\rangle \quad \text{and} \quad \widehat{S}_3(p)|p, \sigma\rangle = \sigma|p, \sigma\rangle \quad (56)$$

and normalization:

$$\langle p, \sigma | q, \tau \rangle = \delta^3(\mathbf{p} - \mathbf{q}) \delta_{\sigma\tau} . \quad (57)$$

while the transformation of a state $|p, \sigma\rangle$ under a general Lorentz transformation Λ reads:

$$\widehat{\Lambda}|p, \sigma\rangle = \sum_{\tau} |\Lambda p, \tau\rangle D_{\tau\sigma}^S([\Lambda p]^{-1} \Lambda[p]) \sqrt{\frac{(\Lambda p)^0}{p^0}} . \quad (58)$$

We are now in a position to develop eq. (51). By using (53) and (54) we get:

$$\Pi(p) = \frac{1}{m} \sum_{\sigma, \sigma'} \langle p, \sigma | \widehat{W} | p, \sigma' \rangle \langle p, \sigma' | \widehat{\rho}_\omega(p) | p, \sigma \rangle = \sum_{i=1}^3 \sum_{\sigma, \sigma'} \langle p, \sigma | \widehat{S}_i(p) | p, \sigma' \rangle \langle p, \sigma' | \widehat{\rho}_\omega(p) | p, \sigma \rangle n_i(p) \quad (59)$$

Two different matrices appear in the above equation. By definition:

$$\langle p, \sigma | \widehat{S}_i(p) | p, \sigma' \rangle = D_{\sigma\sigma'}^S(J_i) \quad (60)$$

are the matrices of the $SU(2)$ generators J_i in the representation labeled by the particle spin S . Furthermore, according to (48):

$$\langle p, \sigma' | \widehat{\rho}_\omega(p) | p, \sigma \rangle = \frac{1}{\frac{1}{(2\pi)^3} \int_V d^3\mathbf{x} e^{i\omega \cdot (\mathbf{x} \times \mathbf{p})/T} \text{tr} D^S(\mathbf{R}_\omega(i\omega/T))} \langle p, \sigma' | \exp[\boldsymbol{\omega} \cdot \widehat{\mathbf{j}}/T] P_V | p, \sigma \rangle \quad (61)$$

As to the rightmost matrix element, we will restore the vector $\boldsymbol{\phi} = i\boldsymbol{\omega}/T$ and calculate it for real $\boldsymbol{\phi}$ (or imaginary $\boldsymbol{\omega}$), then making an analytic continuation to imaginary $\boldsymbol{\phi}$, the same way we did at the end of Sect. II. Therefore:

$$\langle p, \sigma' | \exp[\boldsymbol{\omega} \cdot \widehat{\mathbf{j}}/T] P_V | p, \sigma \rangle = \langle p, \sigma' | \exp[-i\boldsymbol{\phi} \cdot \widehat{\mathbf{j}}] P_V | p, \sigma \rangle = \langle p, \sigma' | \widehat{\mathbf{R}}_\phi(\boldsymbol{\phi}) P_V | p, \sigma \rangle \quad (62)$$

The latter matrix element has been calculated in ref. [3]³:

$$\begin{aligned} \langle p, \sigma' | \widehat{\mathbf{R}}_\phi(\boldsymbol{\phi}) P_V | p, \sigma \rangle &= \int d^3\mathbf{p}' \delta^3(\mathbf{R}_\phi(\boldsymbol{\phi})\mathbf{p}' - \mathbf{p}) \frac{1}{(2\pi)^3} \int_V d^3\mathbf{x} e^{i\mathbf{x} \cdot (\mathbf{p} - \mathbf{p}')} \\ &\quad \times \frac{1}{2} \left(D^S([p]^{-1} \mathbf{R}_\phi(\boldsymbol{\phi}) [p]) + D^S([p]^\dagger \mathbf{R}_\phi(\boldsymbol{\phi}) [p]^\dagger)^{-1} \right)_{\sigma'\sigma} \langle 0 | P_V | 0 \rangle \\ &= \frac{1}{(2\pi)^3} \int_V d^3\mathbf{x} e^{i\mathbf{x} \cdot (\mathbf{p} - \mathbf{R}_\phi(\boldsymbol{\phi})^{-1} \mathbf{p})} \frac{1}{2} \left(D^S([p]^{-1} \mathbf{R}_\phi(\boldsymbol{\phi}) [p]) + D^S([p]^\dagger \mathbf{R}_\phi(\boldsymbol{\phi}) [p]^\dagger)^{-1} \right)_{\sigma'\sigma} \langle 0 | P_V | 0 \rangle \end{aligned} \quad (63)$$

³ It can be easily obtained from eq. (55) in the reference

In this equation, $[p]$ is now meant as an element of the universal covering group of Lorentz group, i.e. $\text{SL}(2, \mathbb{C})$, so that the notation $[p]^\dagger$ becomes meaningful as well as its finite-dimensional representation matrix $D^S([p])$ corresponding to the Lorentz group representation usually labelled as $(S, 0)$ (the $(0, S)$ being the $D^S([p])^{\dagger-1}$). The factor $\langle 0|P_V|0\rangle$ is immaterial and also becomes 1 in the large volume limit as $P_V \rightarrow \mathbb{I}$.

If $\phi \ll 1$, as it was supposed to be at the beginning of this section, then:

$$i\mathbf{x} \cdot (\mathbf{p} - \mathbf{R}_\phi(\phi)^{-1}\mathbf{p}) \simeq -i\phi \cdot (\mathbf{x} \times \mathbf{p})$$

and we can rewrite eq. (63) as:

$$\langle p, \sigma' | \hat{\mathbf{R}}_\phi(\phi) P_V | p, \sigma \rangle \simeq \frac{1}{(2\pi)^3} \int_V d^3x e^{-i\phi \cdot (\mathbf{x} \times \mathbf{p})} \frac{1}{2} \left(D^S([p]^{-1} \mathbf{R}_\phi(\phi) [p]) + D^S([p]^\dagger \mathbf{R}_\phi(\phi) [p]^{\dagger-1}) \right)_{\sigma' \sigma} \quad (64)$$

We can now make an analytical continuation to imaginary ϕ or real ω and obtain, by using (61) and (64):

$$\langle p, \sigma' | \hat{\rho}_\omega(p) | p, \sigma \rangle = \frac{1}{2\text{tr} D^S(\mathbf{R}_\omega(i\omega/T))} \left(D^S([p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S([p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right)_{\sigma' \sigma} \quad (65)$$

This matrix is hermitian and has trace equal to 1, as required for a good density matrix. Hermiticity can be shown by taking advantage of a remarkable feature of $\text{SL}(2, \mathbb{C})$ representation [11]:

$$D^S(A^\dagger) = D^S(A)^\dagger$$

and taking into account that $D^S(\mathbf{R}_\omega(i\omega/T))$ is hermitian.

Plugging (65) into (59), using (60) and carrying out the momentum integration, the final expression of the polarization four-vector for a particle of momentum p in the observer frame is finally achieved:

$$\begin{aligned} \Pi(p)^\mu &= \frac{1}{2\text{tr} D^S(\mathbf{R}_\omega(i\omega/T))} \sum_{i=1}^3 \text{tr} \left[D^S(J_i[p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S(J_i[p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right] n_i(p) \\ &= \frac{1}{2\text{tr} D^S(\mathbf{R}_\omega(i\omega/T))} \sum_{i=1}^3 \text{tr} \left[D^S(J_i[p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S(J_i[p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right] ([p] e_i)^\mu \end{aligned} \quad (66)$$

The proper polarization four-vector, in the particle rest-frame, has components $([p]^{-1} \Pi(p))^\mu$, that is:

$$\Pi_0(p)^\mu = \frac{1}{2\text{tr} D^S(\mathbf{R}_\omega(i\omega/T))} \sum_{i=1}^3 \text{tr} \left[D^S(J_i[p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S(J_i[p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right] \delta_i^\mu \quad (67)$$

which has vanishing time component, as required.

The equations (66) and (67) are the general analytical expression of the polarization for a particle with spin S . We will now develop them for the most interesting cases of spin 1/2 and spin 1. Before doing this, we first observe that the Lorentz transformation $[p]$ to calculate the polarization can be chosen arbitrarily. In fact, if $[p]' \neq [p]$ also transforms the timelike vector $p_0 = (m, 0, 0, 0)$ into p , then $[p]^{-1}[p]'$ is a pure rotation \mathbf{R} as it leaves p_0 invariant. Hence, the polarization four-vector $\Pi(p)$ defined with $[p]'$ becomes:

$$\begin{aligned} \Pi(p) &\propto \sum_{i=1}^3 \frac{1}{2} \text{tr} \left[D^S(J_i[p]'^{-1} \mathbf{R}_\omega(i\omega/T) [p]') + D^S(J_i[p]'^\dagger \mathbf{R}_\omega(i\omega/T) [p]'^{\dagger-1}) \right] ([p]' e_i) \\ &= \sum_{i=1}^3 \frac{1}{2} \text{tr} \left[D^S(J_i \mathbf{R}^{-1} [p]^{-1} \mathbf{R}_\omega(i\omega/T) [p] \mathbf{R}) + D^S(J_i \mathbf{R}^{-1} [p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1} \mathbf{R}) \right] ([p] \mathbf{R} e_i) \\ &= \sum_{i=1}^3 \frac{1}{2} \text{tr} \left[D^S(\mathbf{R} J_i \mathbf{R}^{-1} [p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S(\mathbf{R} J_i \mathbf{R}^{-1} [p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right] ([p] \mathbf{R} e_i) \\ &= \sum_{i=1}^3 \frac{1}{2} \text{tr} \left[D^S(J_{\mathbf{R} e_i} [p]^{-1} \mathbf{R}_\omega(i\omega/T) [p]) + D^S(J_{\mathbf{R} e_i} [p]^\dagger \mathbf{R}_\omega(i\omega/T) [p]^{\dagger-1}) \right] ([p] \mathbf{R} e_i) \end{aligned} \quad (68)$$

where we used the unitarity of \mathbf{R} . The last expression in (68) is apparently equal to the one in (66).

A. Spin S=1/2

Choosing for $[p]$ a pure Lorentz boost, we have [11]:

$$D^{1/2}([p]) = D^{1/2}([p])^\dagger = \frac{m + \varepsilon + \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \quad (69)$$

and

$$D^{1/2}([p])^{-1} = D^{1/2}([p])^{\dagger-1} = \frac{m + \varepsilon - \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \quad (70)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. Therefore:

$$\text{tr} \left[D^{1/2}(J_i[p]^{-1} \mathbf{R}_{\hat{\omega}}(i\omega/T)[p]) \right] = \text{tr} \left[\frac{\sigma_i}{2} \frac{m + \varepsilon - \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \mathbf{R}_{\hat{\omega}}(i\omega/T) \frac{m + \varepsilon + \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \right] \quad (71)$$

whereas, because of (69) and (70):

$$\text{tr} \left[D^{1/2}(J_i[p]^\dagger \mathbf{R}_{\hat{\omega}}(i\omega/T)[p]^{\dagger-1}) \right] = \text{tr} \left[\frac{\sigma_i}{2} \frac{m + \varepsilon + \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \mathbf{R}_{\hat{\omega}}(i\omega/T) \frac{m + \varepsilon - \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \right] \quad (72)$$

i.e. it is obtained from (71) by reflecting \mathbf{p} . Since:

$$D^{1/2}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) = \mathbf{I} \cos \frac{i\omega}{2T} - i\boldsymbol{\sigma} \cdot \hat{\omega} \sin \frac{i\omega}{2T} = \mathbf{I} \cosh \frac{\omega}{2T} + \boldsymbol{\sigma} \cdot \hat{\omega} \sinh \frac{\omega}{2T} \quad (73)$$

we have, for (71)

$$\begin{aligned} & \frac{m + \varepsilon - \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \left(\mathbf{I} \cosh \frac{\omega}{2T} + \boldsymbol{\sigma} \cdot \hat{\omega} \sinh \frac{\omega}{2T} \right) \frac{m + \varepsilon + \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(m + \varepsilon)}} \\ &= \mathbf{I} \cosh \frac{\omega}{2T} + \sinh \frac{\omega}{2T} \left[\frac{m + \varepsilon}{2m} \boldsymbol{\sigma} \cdot \hat{\omega} + \frac{i}{m} \boldsymbol{\sigma} \cdot (\hat{\omega} \times \mathbf{p}) - \frac{\hat{\omega} \cdot \mathbf{p} \boldsymbol{\sigma} \cdot \mathbf{p} - \boldsymbol{\sigma} \cdot (\mathbf{p} \times (\hat{\omega} \times \mathbf{p}))}{2m(\varepsilon + m)} \right] \end{aligned} \quad (74)$$

Now we have to take the sum of the two traces (71) and (72) implying that all terms which change sign in a reflection of the momentum \mathbf{p} in (74) vanish. Therefore, we are left with:

$$\begin{aligned} & 2 \text{tr} \left\{ \frac{\sigma_i}{2} \left[\mathbf{I} \cosh \frac{\omega}{2T} + \sinh \frac{\omega}{2T} \left[\frac{m + \varepsilon}{2m} \boldsymbol{\sigma} \cdot \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \boldsymbol{\sigma} \cdot \mathbf{p} - \boldsymbol{\sigma} \cdot (\mathbf{p} \times (\hat{\omega} \times \mathbf{p}))}{2m(\varepsilon + m)} \right] \right] \right\} \\ &= \sinh \frac{\omega}{2T} \left[\frac{m + \varepsilon}{m} \hat{\omega}_i - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p}_i - (\mathbf{p} \times (\hat{\omega} \times \mathbf{p}))_i}{m(\varepsilon + m)} \right] \end{aligned} \quad (75)$$

Taking into account that:

$$\text{tr} D^{1/2}(\mathbf{R}_{\hat{\omega}}(i\omega/T)) = 2 \cosh \frac{\omega}{2T} \quad (76)$$

we can obtain the polarization vector in the rest-frame putting (75) and (76) into eq. (67):

$$\begin{aligned} \boldsymbol{\Pi}_0 &= \frac{1}{4} \tanh \frac{\omega}{2T} \left[\frac{m + \varepsilon}{m} \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p} - \mathbf{p} \times (\hat{\omega} \times \mathbf{p})}{m(\varepsilon + m)} \right] \\ &= \frac{1}{4} \tanh \frac{\omega}{2T} \left[\frac{m + \varepsilon}{m} \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p} - \mathbf{p}^2 \hat{\omega} + \hat{\omega} \cdot \mathbf{p} \mathbf{p}}{m(\varepsilon + m)} \right] \\ &= \frac{1}{2} \tanh \frac{\omega}{2T} \left[\frac{\varepsilon}{m} \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p}}{m(\varepsilon + m)} \right] \end{aligned} \quad (77)$$

Therefore, unlike in the non-relativistic case, the polarization vector has a component along particle momentum in the observer frame. This effect is owing to the vector nature of the polarization; indeed, the components of polarization in the observer frame (66) can be obtained with a general Lorentz boost [12]:

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_0 + \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} \cdot \boldsymbol{\Pi}_0 \boldsymbol{\beta} \quad \Pi^0 = \gamma \boldsymbol{\beta} \cdot \boldsymbol{\Pi}_0 \quad (78)$$

with $\beta = \mathbf{p}/\varepsilon$ and $\gamma = \varepsilon/m$. Therefore, by using (77), eq. (78) becomes:

$$\mathbf{\Pi} = \frac{1}{2} \tanh \frac{\omega}{2T} \frac{\varepsilon}{m} \hat{\omega} \quad \Pi^0 = \frac{1}{2} \tanh \frac{\omega}{2T} \frac{\hat{\omega} \cdot \mathbf{p}}{m} \quad (79)$$

Hence, the polarization three-vector is aligned with the angular velocity in the *observer frame* and not in the particle frame. Yet, there is also a time component which vanishes in the non-relativistic limit $\mathbf{p}/m \rightarrow 0$, where the (79) correctly yields (49). The remarkable difference with respect to the non-relativistic case (49) is that polarization now depends on momentum. The longitudinal component, along \mathbf{p} , is:

$$\mathbf{\Pi}_0 \cdot \hat{\mathbf{p}} = \frac{1}{2} \tanh \frac{\omega}{2T} \hat{\mathbf{p}} \cdot \hat{\omega} \quad (80)$$

while the component along the rotation axis turns out to be:

$$\mathbf{\Pi}_0 \cdot \hat{\omega} = \frac{1}{2} \tanh \frac{\omega}{2T} \left[\frac{\varepsilon}{m} - \frac{\mathbf{p}^2 (\hat{\omega} \cdot \hat{\mathbf{p}})^2}{m(\varepsilon + m)} \right] \quad (81)$$

which shows a very interesting feature: the polarization is maximal for particles with momentum orthogonal to the rotation axis. Furthermore, the polarization increases with energy, being proportional to ε/m . Of course this behaviour cannot go on indefinitely because polarization cannot exceed 1/2 in any direction; since $\omega/T \ll 1$, it is seen from eq. (81) that something must happen when $\varepsilon \sim 2mT/\omega$. Indeed, we have pointed out in Sect. IV that when momentum is of the order of J/R , the saddle point expansion of partition function at fixed J can no longer be independent of \mathbf{p} , and ω becomes in fact a function of p , that is $\omega(p)$, for the spectrum (32) expansion. This has some impact on the polarization vector: at some large momentum the dependence of ω on p should restore the natural 1/2 bound.

B. Spin S=1

The calculation of the polarization for massive spin 1 particles is more involved than for spin 1/2 and we have carried it out by performing explicitly the multiplication of matrices of $\text{SL}(2, \mathbb{C})$ D^1 representation. Since eventually one has to calculate traces, the choice of the basis for the matrices is arbitrary and we have written $[p] = \mathbf{R}_3(\varphi)\mathbf{R}_2(\theta)\mathbf{L}_3(\xi)$, being $\cosh \xi = \varepsilon/m$, in the cartesian basis for the D^1 representation space:

$$\begin{aligned} D^1(\mathbf{R}_3(\varphi)) &= \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} & D^1(\mathbf{R}_2(\theta)) &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ D^1(\mathbf{L}_3(\xi)) &= \exp[-i\xi K_3] = \exp[\xi J_3] = D^1(\mathbf{R}_3(i\xi)) = \begin{pmatrix} \cosh \xi & -i \sinh \xi & 0 \\ i \sinh \xi & \cosh \xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ D^1(\mathbf{R}_{\hat{\omega}}(i\omega/T)) &= \exp[\hat{\omega}/T \cdot D^1(\mathbf{J})] & D^1(J_i)_{jk} &= -i\epsilon_{ijk} \end{aligned} \quad (82)$$

Multiplying the matrices in eq. (82) and calculating the traces, according to eq. (67), one gets:

$$\mathbf{\Pi}_0 = \frac{2 \sinh(\omega/T)}{2 \cosh(\omega/T) + 1} \left[\frac{\varepsilon}{m} \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p}}{m(\varepsilon + m)} \right] \quad (83)$$

where the denominator $2 \cosh(\omega/T) + 1$ is the trace of the matrix $D^1(\mathbf{R}_{\hat{\omega}}(i\omega/T))$. Therefore, the kinematical structure of the polarization vector is the same as in the spin 1/2 case, which is a reasonable outcome; in fact, the properties of Lorentz transformation of a polarization vector should not depend on the particle spin itself.

In the case of spin 1 it is also interesting to calculate the fraction of the 00 component of the density matrix (65). The calculation can be done quickly by noting that:

$$\rho_{\omega 00}(p) = \text{tr} P_3 \hat{\rho} = \sum_{\sigma} \langle p, \sigma | P_3 \hat{\rho}_{\omega}(p) | p, \sigma \rangle \quad (84)$$

P_3 being the projector onto the state $|p, 0\rangle$. Written in the cartesian basis the matrix corresponding to P_3 is simply:

$$P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (85)$$

because of the choice of z axis as quantization axis, according to (56) Therefore, by plugging eq. (65) into eq. (84):

$$\rho_{\omega 00}(p) = \frac{1}{2\text{tr}D^1(\mathbf{R}_{\hat{\omega}}(i\omega/T))} \text{tr} [P_3 D^1([p]^{-1} \mathbf{R}_{\hat{\omega}}(i\omega/T)[p]) + P_3 D^1([p]^\dagger \mathbf{R}_{\hat{\omega}}(i\omega/T)[p]^\dagger)^{-1}] \quad (86)$$

yielding:

$$\rho_{\omega 00}(p) = \frac{1}{2 \cosh(\omega/T) + 1} \left[\cosh(\omega/T) + \frac{(\mathbf{p} \cdot \hat{\omega})^2}{p^2 \omega^2} (1 - \cosh(\omega/T)) \right] \quad (87)$$

For small ω/T , we have:

$$\rho_{\omega 00}(p) \simeq \frac{1}{3} + \frac{1 - 3(\hat{\mathbf{p}} \cdot \hat{\omega})^2}{18} \frac{\omega^2}{T^2} \quad (88)$$

C. Discussion

In both spin 1/2 and spin 1 cases, we have seen that the proper polarization vector has the same structure, so that we can easily generalize the above results to any spin:

$$\mathbf{\Pi}_0 = \frac{\sum_{n=-S}^S n e^{n\omega/T}}{\sum_{n=-S}^S e^{n\omega/T}} \left[\frac{\varepsilon}{m} \hat{\omega} - \frac{\hat{\omega} \cdot \mathbf{p} \mathbf{p}}{m(\varepsilon + m)} \right] \quad (89)$$

This non-vanishing polarization means that spin states are not evenly populated in an equilibrated thermodynamical system with finite, macroscopic, angular momentum. Of course reaching equilibrium for the spin degrees of freedom implies that a small interaction should exist involving particle spin, e.g. a spin-orbit coupling. Yet, for the ideal relativistic gas, this interaction is assumed to be negligible in comparison with the pure kinematical effect of angular momentum conservation. This can be rephrased in the rotating frame as follows: the actual interaction involving spin is negligible with respect to the spin-rotation coupling. Since a macroscopic thermodynamical system can be conceptually divided into elementary fluid cells, and each cell participating in the rigid rotation is an accelerated system, for our result to be consistent with locality, one has to conclude that, in general, acceleration involves a polarization expressed by (89) where ω is to be interpreted as a local vector field involving local quantities such as velocity and acceleration of the cell.

Acknowledgments

We are grateful for interesting discussions to R. Jaffe, G. Longhi, L. Lusanna, K. Rajagopal, H. Satz, D. Seminara. We thank Galileo Galilei Institute for hospitality.

APPENDIX A - ENTROPY FOR A SPINNING SYSTEM

The entropy of a system with finite angular momentum \mathbf{J} can be calculated easily from the expression of the probability of a state with energy E , vanishing momentum, charge Q and fixed \mathbf{J} in the grand-canonical ensemble:

$$p = \frac{1}{Z_J} \exp[-E/T + \mu Q/T] \quad (90)$$

so that, by using (14)

$$S = - \sum_{\text{states with fixed } \mathbf{J}} p \log p = \frac{\langle E \rangle}{T} - \frac{\mu \langle Q \rangle}{T} + \log Z_J = \frac{U}{T} - \frac{\mu \langle Q \rangle}{T} - \frac{\omega \cdot \mathbf{J}}{T} + \log Z_\omega \quad (91)$$

which is the known expression of entropy for a rotating system. The logarithm of the partition function can then be identified with the integral of the pressure:

$$\log Z_\omega = \frac{1}{T} \int_V d^3x p(\mathbf{x}) \quad (92)$$

where, unlike in familiar cases, the pressure is not uniform due to rotation. From the entropy expression:

$$TS = U - \mu\langle Q \rangle - \boldsymbol{\omega} \cdot \mathbf{J} + \int_V d^3x p(\mathbf{x}) \quad (93)$$

we can derive the relation [5]

$$\left. \frac{\partial S}{\partial \mathbf{J}} \right|_{V,U,Q} = -\frac{\boldsymbol{\omega}}{T} \quad (94)$$

APPENDIX B - EQUILIBRIUM CONFIGURATION OF A RELATIVISTIC ROTATING SYSTEM

We will now show that a relativistic macroscopic system with finite angular momentum at equilibrium must be rigidly rotating by generalizing an argument by Landau [5] for non-relativistic system.

Let us consider a generic isolated hydrodynamical system with velocities \mathbf{v}_i i being the label of hydrodynamical cells. In order to find the equilibrium configuration the entropy should be maximized with the constraint of energy, momentum and angular momentum conservation. Therefore, we have to find the extremum points of:

$$\sum_i S_i - \beta(\sum_i E_i - E_0) + \boldsymbol{\beta} \cdot \sum_i \mathbf{P}_i + \boldsymbol{\omega} \cdot (\sum_i \mathbf{x}_i \times \mathbf{P}_i - \mathbf{J}) \quad (95)$$

where we have taken the total momentum vanishing, i.e. we are working in the system's rest frame; E_0 is the total energy and \mathbf{J} the total angular momentum; β , $\boldsymbol{\beta}$ and $\boldsymbol{\omega}/T$ are Lagrange multipliers enforcing the conservation laws. Since the entropy is a relativistic invariant, it can only depend on the mass of the cell, i.e. $S_i = S_i(\sqrt{E_i^2 - \mathbf{P}_i^2})$. In order to find the equilibrium configuration, one has to maximize (95) with respect to each \mathbf{P}_i and E_i . Thus:

$$\frac{\partial S_i}{\partial E_i} = \frac{E_i}{M_i} \frac{\partial S_i}{\partial M_i} = \beta \quad \forall i \quad (96)$$

and:

$$-\frac{\partial S_i}{\partial \mathbf{P}_i} = \frac{\mathbf{P}_i}{M_i} \frac{\partial S_i}{\partial M_i} = \boldsymbol{\beta} + \boldsymbol{\omega} \times \mathbf{x}_i \quad \forall i \quad (97)$$

Taking into account that $\partial S_i / \partial M_i$ is the inverse of the proper temperature T_i of the i th cell by definition, eq. (96) implies that:

$$\frac{\gamma_i}{T_i} = \text{constant} = \beta \equiv \frac{1}{T} \quad (98)$$

where T is defined the *global* temperature of the system. Plugging (98) into (97) we get:

$$\frac{\gamma_i \mathbf{v}_i}{T_i} = \frac{\mathbf{v}_i}{T} = \boldsymbol{\beta} + \frac{\boldsymbol{\omega} \times \mathbf{x}_i}{T} \quad (99)$$

For the total momentum to vanish, the vector $\boldsymbol{\beta}$ should be 0 as well and we are left with:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{x}_i \quad (100)$$

that is a rigid rotation around the axis $\boldsymbol{\omega}$. To show that $T = 1/\beta$ is in fact the global temperature of the system, one has to consider that at equilibrium, using formula (91) in Appendix A for the entropy of the cell i :

$$S \equiv \sum_i S_i = \sum_i \frac{M_i}{T_i} + \dots = \sum_i \frac{\gamma_i M_i}{T} + \dots = \sum_i \frac{E_i}{T} = \frac{M}{T} + \dots \quad (101)$$

M being the mass of the system, because $\mathbf{P} = 0$. Therefore:

$$\frac{\partial S}{\partial M} = \frac{1}{T}$$

showing that T is the actual temperature of the system. The relation (98) can be rewritten as, by using (100):

$$T = T_i \sqrt{1 - \|\boldsymbol{\omega} \times \mathbf{x}_i\|^2} \quad (102)$$

in accordance with eq. (11). The involved physical meaning is as follows: the temperature measured by a thermometer at rest in the observer frame, is *lower* than that measured by a comoving thermometer, i.e. at rest in the cell frame. This is a pure relativistic effect which has no correspondance in classical thermodynamics.

The above argument can be extended to include the intrinsic angular momentum contribution \mathbf{J}_i of each cell to \mathbf{J} . Eq. (95) now reads:

$$\sum_i S_i (\sqrt{E_i^2 - \mathbf{P}_i^2}, \mathbf{J}_i) - \beta (\sum_i E_i - E_0) + \beta \cdot \sum_i \mathbf{P}_i + \beta \boldsymbol{\omega} \cdot (\sum_i \mathbf{x}_i \times \mathbf{P}_i + \mathbf{J}_i - \mathbf{J}) \quad (103)$$

The conclusions are similar, with the additional condition:

$$\frac{\partial S_i}{\partial \mathbf{J}_i} = -\frac{\boldsymbol{\omega}}{T}$$

in accordance with eq. (94).

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